Math 279 Lecture 1 Notes

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August 26, 2021

1 A Motivating Example for Studying Stochastic PDEs

1.1 Fluids: an example of what a stochastic PDE looks like

Here is an analogy. When you see a line of ants, you may think that the line is relatively straight, so you write down an equation that describes the motion. If you increase the precision of your model, you may see that the ants actually move with some random fluctuations, so you add some randomness to your model. The more precision you require, the more you realize that the ants are not moving in a straight line at all and are instead constantly bumping into each other, exchanging information. This is how stochastic PDEs are.

Imagine that we have a fluid for which the velocity of fluid particles are known, say $u(x,t)$. As a simple model for the fluid particle, we write

$$\frac{dx}{dt} = u(x,t).$$

This is an ODE which, as a first approximation, gives us a good idea of a model for what is happening. To take into account the thermal fluctuation of the fluid, we may write

$$\frac{dx}{dt} = \underbrace{u(x,t)}_{\text{vector field}} + \underbrace{\sigma(x,t)}_{\text{matrix}} \eta(t), \tag{1}$$

with η representing "white noise" (to be formally defined later) and $\sigma(x,t)$ measuring the strength of the fluctuation at (x,t). Here, η is a Gaussian process with $\mathbb{E}[\eta(t)] = 0$ and $\mathbb{E}[\eta(t)\eta(s)] = \delta_0(t-s)$, where δ_0 is the Dirac delta "function" at 0.

In reality, u itself solves some PDE, and in the case of a (viscous) incompressible fluid, we have

$$u_t + (u \cdot \nabla)u + \underbrace{\nabla P}_{\text{pressure}} = \nu \Delta u + \underbrace{f}_{\text{force}}$$

$$\nabla \cdot u = 0,$$

where for simplicity we assume $\sigma = \sqrt{2\nu}I$. We have a system of 4 equations with 3 unknowns (the function u), so we need to solve this equation for the pair (u, P). A natural model example for f is that $f = f(x,t)$ is "white noise" in (x,t) (sometimes, we assume f is white in t and "colored" in x).

1.2 Regularity issues with white noise

Going back to the previous equation (1), how can we make sense of this equation? The problem is that "white noise" cannot be realized as a function. A solution to (1) is an example of a diffusion.[1] Observe that if $u = 0$ and $\sigma = 1$, then $\frac{dx}{dt} = \eta$. As it turns out, $x(t) = x(0) + B(t)$, where B is a standard Brownian motion.[2] It is well-known that Brownian motion can be realized as a continuous function, in fact $B \in \mathcal{C}^{1/2-}$. Here, we write $\mathcal{C}^\alpha$ as the space of Hölder continuous functions of exponent α and $\mathcal{C}^{\alpha-} = \bigcap_{\beta<\alpha} \mathcal{C}^\beta$. In fact, $\eta = \dot{B} \in \mathcal{C}^{-1/2-}$. By $f \in \mathcal{C}^\beta$ for $\beta < 0$, we mean $f = \dot{g}$ with $g \in \mathcal{C}^{\beta+1}$ (we will give a more robust definition of this later).

Going back to $\dot{x} = u(x,t) + \sigma(x,t)\eta(t)$, we expect this to have a solution $x(\cdot) \in \mathcal{C}^{1/2-}$. To make sense of this, we write

$$x(t) = x(0) + \int_0^t u(x(s), s)\, ds + \int_0^t \sigma(x(s), s)\, \underbrace{\eta(s)\, ds}_{dB(s)}.$$

We face the following difficulty:

$$\eta(\varphi) = \int \eta(s)\varphi(s)\, ds = \int \dot{B}(s)\varphi(s)\, ds \overset{\text{IBP}}{=} -\int B(s)\dot{\varphi}(s)\, ds,$$

where φ is smooth with compact support. The problem is that f is not $\mathcal{C}^1$, only $\mathcal{C}^{1/2-}$. This calls for studying $\int_0^t g\, df$ with f, g continuous functions. This problem has a rich history that we now review:

1. In fact, Riemann and Steiltjes defined the integral $\int_0^t g\, df$ as

$$\int_0^t g\, df = \lim_{n\to\infty} \sum_{i=0}^{2^n} g(s_i)(f(t_{i+1}) - f(t_i)) \tag{2}$$

   with $s_i \in [t_i, t_{i+1}]$, where the t_i form a mesh with 2^n points. It turns out that this equation converges (no matter what we choose for s_i) if g is continuous ($g \in \mathcal{C}^0$) and $f \in \text{BV}$ is of bounded variation. Recall that $f \in \text{BV}$ means $\|f\|_{\text{BV}} < \infty$, where $\|f\|_{\text{BV}} = \sup_{0<t_1<\cdots<t_k<t} \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)|$. In particular, if $g \in \mathcal{C}^0$ and $f \in \mathcal{C}^1$, then $\int_0^t f\, dg$ can be defined.

[1] Diffusions were first described by Kolmogorov in the early 30s and later described by Paul Lévy and Itô.

[2] The moral here is to still differentiate things that are not differentiable. Don't let that stop you.

2. Lebesgue theory allows us to interpret $\int_0^t f\,dg$ as $\int_0^t f\,d\mu$, where $\mu = g'$ in a weak sense:

$$\int \varphi\,d\mu = g'(\varphi) = -\int \varphi' g\,dt$$

for all smooth φ. In this picture, $f \in \mathrm{BV} \iff f'$ can be realized as a measure.

3. So far, we know how to define $\int g\,df$ with $g \in \mathcal{C}^0, f \in \mathrm{BV}$. But we can also make sense of it if $g \in \mathrm{BV}, f \in \mathcal{C}^0$ by declaring $\int_0^t g\,df = g(t)f(t) - g(0)f(0) - \int_0^t f\,dg$.

4. Young observed that equation (2) still works if $f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta$ with $\alpha + \beta > 1$. In fact, (2) works even when $f \in \mathrm{BV}_{1/\alpha}, g \in \mathrm{BV}_{1/\beta}$, where

$$\|f\|_{\mathrm{BV}_p} = \sup_{0 \le t_1 < \cdots < t_h \le t} \sum_i |f(t_{i+1}) - f(t_i)|^p$$

for $p \geq 1$. Observe that $\mathrm{BV}_{1/\alpha} \supsetneq \mathcal{C}^\alpha$. Moreover, Young proved that $h(t) = \int_0^t g\,df$ satisfies the following bound:

$$|h(t) - h(s) - g(s)(f(t) - f(s))| \le c|t - s|^{\alpha+\beta} \tag{3}$$

where c is a constant depending on $\|f\|_{\mathcal{C}^\alpha}$ and $\|g\|_{\mathcal{C}^\alpha}$. In fact, h can be uniquely specified as the only function for which $h(0) = 0$, and h satisfies (for some constant c) (3). If $h, \widetilde{h}$ are two solutions, then $k = h - \widetilde{h}$ satisfies $|k(t) - k(s)| \le c|t - s|^{\alpha+\beta}$.

1.3 Ways of defining the stochastic integral with irregular functions

Going back to our integral $\int_0^t \sigma(x(s), s)\,dB(s)$, Young's theory does not apply because both $\sigma(x(s), s)$ and $B(s)$ are both in $\mathcal{C}^{1/2-}$. As an example, consider $\int_0^t F(B(s))\,dB(s)$ for $F \in \mathcal{C}^1$. In fact, the approximation in (2) may fail in two ways. Either the limit does not exist or the limit exists but depends on the choice of s_i! Some popular choices of limits in probability theory are:

Example 1.1. Itô defined the integral

$$M(t) = \int_0^t F(B(s))\,dB(s) = \lim_{n\to\infty} \sum_{i=0}^{2^n - 1} F(B(t_i))(B(t_{i+1}) - B(t_i)).$$

The advantage is that the outcome $M(t)$ is a martingale.

Here is another choice:

Example 1.2. Statonovich defined the approximation by replacing $F(B(t_i))$ with

$$\frac{F(B(t_i)) + F(B(t_{i+1}))}{2}.$$

There is also a "backward" way, where we choose $F(B(t_{i+1}))$ instead. Next time, we will discuss the drawbacks of Itô calculus and introduce rough path theory.